

Weights sharing the same eigenvalue

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Dedicated to Professor Wataru Takahashi, with esteem and friendship, on his seventieth birthday

Abstract: Here is the simplest particular case of our main result: let $f : \mathbf{R} \rightarrow \mathbf{R}$ be a function of class C^1 , with $\sup_{\mathbf{R}} f' > 0$, such that

$$\lim_{|\xi| \rightarrow +\infty} \frac{f(\xi)}{\xi} = 0 .$$

Then, for each $\lambda > \frac{\pi^2}{\sup_{\mathbf{R}} f'}$, the set of all $u \in H_0^1(]0, 1[)$ for which the problem

$$\begin{cases} -v'' = \lambda f'(u(x))v & \text{in }]0, 1[\\ v(0) = v(1) = 0 \end{cases}$$

has a non-zero solution is closed and not σ -compact in $H_0^1(]0, 1[)$.

Key words: Eigenvalue; weight; Dirichlet problem; σ -compact.

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1. Introduction

Let $\Omega \subset \mathbf{R}^n$ be a smooth bounded domain. We consider the Sobolev space $H_0^1(\Omega)$ endowed with the scalar product

$$\langle u, v \rangle = \int_{\Omega} \nabla u(x) \nabla v(x) dx$$

and the induced norm

$$\|u\| = \left(\int_{\Omega} |\nabla u(x)|^2 dx \right)^{\frac{1}{2}} .$$

We are interested in pairs (λ, β) , where λ is a positive number and β is a measurable function, such that the linear problem

$$\begin{cases} -\Delta v = \lambda \beta(x)v & \text{in } \Omega \\ v|_{\partial\Omega} = 0 \end{cases}$$

has a non-zero weak solution, that is to say a $v \in H_0^1(\Omega) \setminus \{0\}$ such that

$$\int_{\Omega} \nabla v(x) \nabla w(x) dx = \lambda \int_{\Omega} \beta(x)v(x)w(x) dx$$

for all $w \in H_0^1(\Omega)$.

If this happens, we say that λ is an eigenvalue related to the weight β .

While, the structure of the set of all eigenvalues related to a fixed weight is well understood, it seems that much less is known about the structure of the set of all weights β for which a fixed positive number λ turns out to be an eigenvalue related to β .

In this very short note, we intend to give a contribution along the latter direction.

More precisely, we identify a quite general class of continuous functions $g : \mathbf{R} \rightarrow \mathbf{R}$ such that, for each λ in a suitable interval, the set of all $u \in H_0^1(\Omega)$ for which λ is an eigenvalue related to the weight $g(u(\cdot))$ is closed and not σ -compact in $H_0^1(\Omega)$.

2. Results

Let us recall that a set in a topological space is said to be σ -compact if it is the union of an at most countable family of compact sets.

For each $\alpha \in L^\infty(\Omega) \setminus \{0\}$, with $\alpha \geq 0$, we denote by λ_α the first eigenvalue of the problem

$$\begin{cases} -\Delta v = \lambda_\alpha(x)v & \text{in } \Omega \\ v|_{\partial\Omega} = 0. \end{cases}$$

Let us recall that

$$\lambda_\alpha = \min_{v \in H_0^1(\Omega) \setminus \{0\}} \frac{\|v\|^2}{\int_\Omega \alpha(x)|v(x)|^2 dx}.$$

With the conventions $\frac{1}{+\infty} = 0$, $\frac{1}{0} = +\infty$, here is the statement of the result introduced above:

THEOREM 1. - *Let $f : \mathbf{R} \rightarrow \mathbf{R}$ be a function of class C^1 such that*

$$\max \left\{ 0, 2 \limsup_{|\xi| \rightarrow +\infty} \frac{\int_0^\xi f(t)dt}{\xi^2}, \limsup_{|\xi| \rightarrow +\infty} \frac{f(\xi)}{\xi} \right\} < \sup_{\mathbf{R}} f'.$$

Moreover, if $n \geq 2$, assume that

$$\sup_{\xi \in \mathbf{R}} \frac{|f'(\xi)|}{1 + |\xi|^q} < +\infty$$

for some $q > 0$, with $q < \frac{4}{n-2}$ if $n \geq 3$.

Then, for each $\alpha \in L^\infty(\Omega) \setminus \{0\}$, with $\alpha \geq 0$, and for every λ satisfying

$$\frac{\lambda_\alpha}{\sup_{\mathbf{R}} f'} < \lambda < \frac{\lambda_\alpha}{\max \left\{ 0, 2 \limsup_{|\xi| \rightarrow +\infty} \frac{\int_0^\xi f(t)dt}{\xi^2}, \limsup_{|\xi| \rightarrow +\infty} \frac{f(\xi)}{\xi} \right\}}$$

the set of all $u \in H_0^1(\Omega)$ for which the problem

$$\begin{cases} -\Delta v = \lambda_\alpha(x)f'(u(x))v & \text{in } \Omega \\ v|_{\partial\Omega} = 0 \end{cases}$$

has a non-zero weak solution is closed and not σ -compact in $H_0^1(\Omega)$.

REMARK 1. - It is worth noticing that the linear hull of any closed and not σ -compact set in $H_0^1(\Omega)$ is infinite-dimensional. This comes from the fact that any closed set in a finite-dimensional normed space is σ -compact.

The key tool we use to prove Theorem 1 is Theorem 2 below whose proof, in turn, is entirely based on the following particular case of a result recently established in [1]:

THEOREM A ([1], Theorem 2) *Let $(X, \langle \cdot, \cdot \rangle)$ be an infinite-dimensional real Hilbert space and let $I : X \rightarrow \mathbf{R}$ be a sequentially weakly lower semicontinuous, not convex functional of class C^2 such that I' is closed and $\lim_{\|x\| \rightarrow +\infty} (I(x) + \langle z, x \rangle) = +\infty$ for all $z \in X$.*

Then, the set

$$\{x \in X : I''(x) \text{ is not invertible}\}$$

is closed and not σ -compact.

THEOREM 2. - *Let $(X, \langle \cdot, \cdot \rangle)$ be an infinite-dimensional real Hilbert space, and let $J : X \rightarrow \mathbf{R}$ be a functional of class C^2 , with compact derivative. For each $\lambda \in \mathbf{R}$, put*

$$A_\lambda = \{x \in X : y = \lambda J''(x)(y) \text{ for some } y \in X \setminus \{0\}\} .$$

Assume that

$$\max \left\{ 0, 2 \limsup_{\|x\| \rightarrow +\infty} \frac{J(x)}{\|x\|^2}, \limsup_{\|x\| \rightarrow +\infty} \frac{\langle J'(x), x \rangle}{\|x\|^2} \right\} < \sup_{(x,y) \in X \times (X \setminus \{0\})} \frac{\langle J''(x)(y), y \rangle}{\|y\|^2} .$$

Then, for every λ satisfying

$$\inf_{\{(x,y) \in X \times X : \langle J''(x)(y), y \rangle > 0\}} \frac{\|y\|^2}{\langle J''(x)(y), y \rangle} < \lambda < \frac{1}{\max \left\{ 0, 2 \limsup_{\|x\| \rightarrow +\infty} \frac{J(x)}{\|x\|^2}, \limsup_{\|x\| \rightarrow +\infty} \frac{\langle J'(x), x \rangle}{\|x\|^2} \right\}} ,$$

the set A_λ is closed and not σ -compact .

PROOF. Fix λ satisfying

$$\inf_{\{(x,y) \in X \times X : \langle J''(x)(y), y \rangle > 0\}} \frac{\|y\|^2}{\langle J''(x)(y), y \rangle} < \lambda < \frac{1}{\max \left\{ 0, 2 \limsup_{\|x\| \rightarrow +\infty} \frac{J(x)}{\|x\|^2}, \limsup_{\|x\| \rightarrow +\infty} \frac{\langle J'(x), x \rangle}{\|x\|^2} \right\}} .$$

For each $x \in X$, put

$$I_\lambda(x) = \frac{1}{2} \|x\|^2 - \lambda J(x) .$$

Clearly, for some $(x, y) \in X \times X$, with $\langle J''(x)(y), y \rangle > 0$, we have

$$\langle y - \lambda J''(x)(y), y \rangle < 0$$

and so, since

$$I_\lambda''(x)(y) = y - \lambda J''(x)(y) ,$$

by a classical characterization (Theorem 2.1.11 of [2]), the functional I_λ is not convex. Now, let us show that

$$\lim_{\|x\| \rightarrow +\infty} \|x - \lambda J'(x)\| = +\infty . \quad (1)$$

Indeed, for each $x \in X \setminus \{0\}$, we have

$$\|x - \lambda J'(x)\| = \sup_{\|y\|=1} \langle x + \lambda J'(x), y \rangle \geq \left\langle x - \lambda J'(x), \frac{x}{\|x\|} \right\rangle \geq \|x\| \left(1 - \lambda \frac{\langle J'(x), x \rangle}{\|x\|^2} \right) . \quad (2)$$

On the other hand, we also have

$$\liminf_{\|x\| \rightarrow +\infty} \left(1 - \lambda \frac{\langle J'(x), x \rangle}{\|x\|^2} \right) = 1 - \lambda \limsup_{\|x\| \rightarrow +\infty} \frac{\langle J'(x), x \rangle}{\|x\|^2} > 0 . \quad (3)$$

So, (1) is a direct consequence of (2) and (3). Furthermore, for each $z \in X$, since

$$I_\lambda(x) + \langle z, x \rangle = \|x\|^2 \left(\frac{1}{2} - \lambda \frac{J(x)}{\|x\|^2} + \frac{\langle z, x \rangle}{\|x\|^2} \right)$$

and

$$\liminf_{\|x\| \rightarrow +\infty} \left(\frac{1}{2} - \lambda \frac{J(x)}{\|x\|^2} + \frac{\langle z, x \rangle}{\|x\|^2} \right) = \frac{1}{2} - \lambda \limsup_{\|x\| \rightarrow +\infty} \frac{J(x)}{\|x\|^2} > 0 ,$$

we have

$$\lim_{\|x\| \rightarrow +\infty} (I_\lambda(x) + \langle z, x \rangle) = +\infty .$$

Since J' is compact, on the one hand, J is sequentially weakly continuous ([4], Corollary 41.9) and, on the other hand, in view of (1), the operator I'_λ is closed ([3], Example 4.43). The compactness of J' also implies that, for each $x \in X$, the operator $J''(x)$ is compact ([3], Proposition 7.33) and so, for each $\lambda \in \mathbf{R}$, the operator $y \rightarrow y - \lambda J''(x)(y)$ is injective if and only if it is surjective ([3], Example 8.16). At this point, the fact that A_λ is closed and not σ -compact follows directly from Theorem A which can be applied to the functional I_λ . \triangle

Proof of Theorem 1. Fix $\alpha \in L^\infty(\Omega) \setminus \{0\}$, with $\alpha \geq 0$. For each $u \in H_0^1(\Omega)$, put

$$J_f(u) = \int_\Omega \alpha(x) F(u(x)) dx ,$$

where

$$F(\xi) = \int_0^\xi f(t) dt .$$

Our assumptions ensure that the functional J_f is of class C^2 in $H_0^1(\Omega)$, and we have

$$\langle J'_f(u), v \rangle = \int_\Omega \alpha(x) f(u(x)) v(x) dx ,$$

$$\langle J''_f(u)(v), w \rangle = \int_\Omega \alpha(x) f'(u(x)) v(x) w(x) dx$$

for all $u, v, w \in H_0^1(\Omega)$. Moreover, J'_f is compact. Fix $\nu > \limsup_{|\xi| \rightarrow +\infty} \frac{F(\xi)}{\xi^2}$. Then, for a suitable constant $c > 0$, we have

$$F(\xi) \leq \nu \xi^2 + c$$

for all $\xi \in \mathbf{R}$. Hence, for each $u \in H_0^1(\Omega)$, we obtain

$$J_f(u) \leq \nu \int_\Omega \alpha(x) |u(x)|^2 dx + c \int_\Omega \alpha(x) dx \leq \nu \lambda_\alpha^{-1} \|u\|^2 + c \int_\Omega \alpha(x) dx .$$

This clearly implies that

$$\limsup_{\|u\| \rightarrow +\infty} \frac{J_f(u)}{\|u\|^2} \leq \lambda_\alpha^{-1} \limsup_{|\xi| \rightarrow +\infty} \frac{F(\xi)}{\xi^2} . \quad (4)$$

In the same way, we obtain

$$\limsup_{\|u\| \rightarrow +\infty} \frac{\langle J'_f(u), u \rangle}{\|u\|^2} \leq \lambda_\alpha^{-1} \limsup_{|\xi| \rightarrow +\infty} \frac{f(\xi)}{\xi} . \quad (5)$$

Now, fix a function $\tilde{v} \in H_0^1(\Omega)$, with $\|\tilde{v}\| = 1$, such that

$$\int_\Omega \alpha(x) |\tilde{v}(x)|^2 dx = \lambda_\alpha^{-1} .$$

Fix also $\epsilon > 0$, $\tilde{\xi} \in \mathbf{R}$, with $f'(\tilde{\xi}) > 0$, and a closed set $C \subseteq \Omega$ so that

$$\int_C \alpha(x) |\tilde{v}(x)|^2 dx > \int_{\Omega} \alpha(x) |\tilde{v}(x)|^2 dx - \epsilon$$

and

$$\int_{\Omega \setminus C} \alpha(x) |\tilde{v}(x)|^2 dx < \frac{\epsilon}{\sup_{[-|\tilde{\xi}|, |\tilde{\xi}|]} |f'|}.$$

Finally, fix a function $\tilde{u} \in H_0^1(\Omega)$ such that

$$\tilde{u}(x) = \tilde{\xi}$$

for all $\xi \in C$ and

$$|\tilde{u}(x)| \leq |\tilde{\xi}|$$

for all $\xi \in \Omega$. Then, we have

$$\begin{aligned} f'(\tilde{\xi}) \left(\int_{\Omega} \alpha(x) |\tilde{v}(x)|^2 dx - \epsilon \right) &< f'(\tilde{\xi}) \int_C \alpha(x) |\tilde{v}(x)|^2 dx = \int_{\Omega} \alpha(x) f'(\tilde{u}(x)) |\tilde{v}(x)|^2 dx - \int_{\Omega \setminus C} \alpha(x) f'(\tilde{u}(x)) |\tilde{v}(x)|^2 dx \\ &\leq \sup_{(u,v) \in H_0^1(\Omega) \times (H_0^1(\Omega) \setminus \{0\})} \frac{\int_{\Omega} \alpha(x) f'(u(x)) |v(x)|^2 dx}{\|v\|^2} + \epsilon. \end{aligned}$$

Since $\tilde{\xi}$ and ϵ are arbitrary, we then infer that

$$\lambda_{\alpha}^{-1} \sup_{\mathbf{R}} f' \leq \sup_{(u,v) \in H_0^1(\Omega) \times (H_0^1(\Omega) \setminus \{0\})} \frac{\int_{\Omega} \alpha(x) f'(u(x)) |v(x)|^2 dx}{\|v\|^2}. \quad (6)$$

Consequently, putting (4), (5) and (6) together, we obtain

$$\begin{aligned} \max \left\{ 0, 2 \limsup_{\|u\| \rightarrow +\infty} \frac{J_f(u)}{\|u\|^2}, \frac{\langle J'_f(u), u \rangle}{\|u\|^2} \right\} &\leq \lambda_{\alpha}^{-1} \max \left\{ 0, 2 \limsup_{|\xi| \rightarrow +\infty} \frac{\int_0^{\xi} f(t) dt}{\xi^2}, \limsup_{|\xi| \rightarrow +\infty} \frac{f(\xi)}{\xi} \right\} \\ &< \lambda_{\alpha}^{-1} \sup_{\mathbf{R}} f' \leq \sup_{(u,v) \in H_0^1(\Omega) \times (H_0^1(\Omega) \setminus \{0\})} \frac{\int_{\Omega} \alpha(x) f'(u(x)) |v(x)|^2 dx}{\|v\|^2}. \end{aligned}$$

Therefore, we can apply Theorem 2 taking $X = H_0^1(\Omega)$ and $J = J_f$. Therefore, for every λ satisfying

$$\frac{\lambda_{\alpha}}{\sup_{\mathbf{R}} f'} < \lambda < \frac{\lambda_{\alpha}}{\max \left\{ 0, 2 \limsup_{|\xi| \rightarrow +\infty} \frac{\int_0^{\xi} f(t) dt}{\xi^2}, \limsup_{|\xi| \rightarrow +\infty} \frac{f(\xi)}{\xi} \right\}},$$

the set A_{λ} (defined in Theorem 2) is closed and not σ -compact in $H_0^1(\Omega)$. But, clearly, a $u \in H_0^1(\Omega)$ belongs to A_{λ} if and only if the problem

$$\begin{cases} -\Delta v = \lambda \alpha(x) f'(u(x)) v & \text{in } \Omega \\ v|_{\partial\Omega} = 0 \end{cases}$$

has a non-zero weak solution, and the proof is complete. \triangle

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